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Abstract.[1] Let $(\mathcal{X},\|\,.\,\|)$ be a Banach space. In general, for a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $(\mathcal{X},\|\,.\,\|)$, its adjoint semigroup $\{T^*(t)\}_{t\geq 0}$ is no longer strongly continuous on the dual space $(\mathcal{X}^*,\|\,.\,\|^*)$. Consider on $\mathcal{X}^*$ the topology of uniform convergence on compact subsets of $(\mathcal{X},\|\,.\,\|)$ denoted by $\mathcal{C}(\mathcal{X}^*,\mathcal{X})$, for which the usual semigroups in literature becomes C_0-semigroups.

The main purpose of this paper is to prove that only a core can be the domain of uniqueness for a C_0-semigroup on $(\mathcal{X}^*,\mathcal{C}(\mathcal{X}^*,\mathcal{X}))$. As application, we show that the generalized Schrödinger operator $\mathcal{A}^V f=\frac{1}{2}\Delta f+b\cdot\nabla f-Vf$, $f\in C_0^\infty(\mathbb{R}^d)$, is $L^\infty\left(\mathbb{R}^d,dx\right)$-unique. Moreover, we prove the $L^1(\mathbb{R}^d,dx)$-uniqueness of weak solution for the Fokker-Planck equation associated with $\mathcal{A}^V$.

[1]**Key Words:** uniqueness of C_0-semigroups; L^∞-uniqueness of generalized Schrödinger operator; L^1-uniqueness of weak solution for the Fokker-Planck equation.

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1 Preliminaries

A complete information on the general theory of strongly continuous semigroups of linear operators can be obtained by consulting the books of Yosida [31], Davies [6], Pazy [23] or Goldstein [12].

In general, for a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space $(\mathcal{X}, \| \,.\, \|)$, it is well known that its adjoint semigroup $\{T^*(t)\}_{t\geq 0}$ is no longer strongly continuous on the dual space $(\mathcal{X}^*, \| \,.\, \|^*)$ with respect to the strong topology of $\mathcal{X}^*$. Without that strong continuity, the theory of semigroups becomes quite complicated and the Hille-Yosida theorem becomes very difficult (see Feller [10], [11], Dynkin [8], Jefferies [14], [15] or Cerrai [5]).

Recentely Wu and Zhang [30] introduced on $\mathcal{X}^*$ a topology for which the usual semigroups in literature becomes C_0-semigroups. That is *the topology of uniform convergence on compact subsets of* $(\mathcal{X}, \| \,.\, \|)$, denoted by $\mathcal{C}(\mathcal{X}^*, \mathcal{X})$.

It is not difficult to prove (see [30, Lemma 1.10, p. 567])

LEMMA 1.1. *Let* $(\mathcal{X}, \| \,.\, \|)$ *be a Banach space. Then* $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$ *is a locally convex space and:*

i) the dual space $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))^*$ *of* $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$ *is* $\mathcal{X}$*;*

ii) any bounded subset of $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$ *is* $\| \,.\, \|^*$*-bounded. And restriction to a* $\| \,.\, \|^*$*-bounded subset of* $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$ *coincides with* $\sigma(\mathcal{X}^*, \mathcal{X})$*;*

iii) $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$ *is complete;*

iv) the topology $\mathcal{C}(\mathcal{X}, \mathcal{X}^*_{\mathcal{C}})$*, where* $\mathcal{X}^*_{\mathcal{C}} = (\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$*, coincides with the* $\| \,.\, \|$*-topology of* $\mathcal{X}$*.*

Moreover, if $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup on $(\mathcal{X}, \| \,.\, \|)$ with generator $\mathcal{L}$, then $\{T^*(t)\}_{t\geq 0}$ is a C_0-semigroup on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$ with generator $\mathcal{L}^*$ (see [30, Theorem

1.4, p.564]). This is a satisfactory variant of Phillips theorem concerning the adjoint of a C_0-semigroup.

Therefore we have all ingredients to consider C_0-semigroups on the locally convex space $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$. In accord to [31, Definiton, p.234], we say that a family $\{T(t)\}_{t\geq 0}$ of linear continuous operators on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$ is a C_0-*semigroup* on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$ if the following properties holds:

(i) $T(0) = I$;

(ii) $T(t+s) = T(t)T(s)$, for all $t, s \geq 0$;

(iii) $\lim_{t\searrow 0} T(t)x = x$, for all $x \in (\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$;

(iv) there exist a number $\omega_0 \in \mathbb{R}$ such that the family $\{e^{-\omega_0 t}T(t)\}_{t\geq 0}$ is equicontinuous.

The *infinitesimal generator* of the C_0-semigroup $\{T(t)\}_{t\geq 0}$ is a linear operator $\mathcal{L}$ defined on the domain

$$\mathcal{D}(\mathcal{L}) = \left\{ x \in \mathcal{X} \;\middle|\; \lim_{t\searrow 0} \frac{T(t)x - x}{t} \text{ exists in } (\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X})) \right\}$$

by

$$\mathcal{L}x = \lim_{t\searrow 0} \frac{T(t)x - x}{t} \quad , \quad \forall x \in \mathcal{D}(\mathcal{L}).$$

We can see that $\mathcal{L}$ is a densely defined and closed operator on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$ and the resolvent $R(\lambda; \mathcal{L}) = (\lambda I - \mathcal{L})^{-1}$, for any $\lambda \in \rho(\mathcal{L})$ (the resolvent set of $\mathcal{L}$) satisfies the equality

$$R(\lambda; \mathcal{L})x = \int_0^\infty e^{-\lambda t} T(t)x \, dt \quad , \quad \forall \lambda > \omega_0 \text{ and } \forall x \in \mathcal{X}^*.$$

Unfortunately, in applications it is difficult to characterise completely the domain of generator $\mathcal{L}$. For this reason, sometimes we need to work on a subspace $\mathcal{D} \subset \mathcal{D}(\mathcal{L})$ dense in $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$ which is called a *core* of generator (see [6, p.7]). More precisely,

DEFINITION 1.2. We say that $\mathcal{D} \subset \mathcal{D}(\mathcal{L})$ is a core of generator $\mathcal{L}$ if $\mathcal{D}$ is dense in $\mathcal{D}(\mathcal{L})$ with respect to the graph topology $\mathcal{C}_{\mathcal{L}}(\mathcal{X}^*, \mathcal{X})$ of $\mathcal{L}$ induced by the topology $\mathcal{C}(\mathcal{X}^*, \mathcal{X})$.

This paper is organized as follows: in the next section by using a Desch-Schappacher perturbation of generator we prove that only a core can be the domain of uniqueness for a C_0-semigroup on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$. This property is well known in the case of C_0-semigroups on Banach spaces (see [2, Theorem 1.33, p.46]), but here we prove it for a C_0-semigroup on the dual of a Banach space. In a forthcoming paper [19] we extend this property to the must difficult case of the dual of a locally convex space.

The Section 3 is devoted to study the $L^\infty\left(\mathbb{R}^d, dx\right)$-uniqueness of generalized Schrödinger operator. Remark that the natural topology for studying this problem is the topology of uniform convergence on compacts subsets of $\left(L^1\left(\mathbb{R}^d, dx\right), \| \,.\, \|_1\right)$ which is denoted by $\mathcal{C}\left(L^\infty, L^1\right)$.

In the first main result of Section 3 we find neccesary and sufficient conditions to show that the one-dimensional operator $\mathcal{A}_1^V f = a(x)f'' + b(x)f' - V(x)f$, $f \in C_0^\infty(x_0, y_0)$, where $-\infty \leq x_0 < y_0 \leq \infty$, is $L^\infty(x_0, y_0)$-unique.
In the second important result, by comparison with the one-dimensional case, we prove that the multidimensional generalized Schrödinger operator $\mathcal{A}^V f = \frac{1}{2}\Delta f + b\cdot\nabla f - Vf$, $f \in C_0^\infty(\mathbb{R}^d)$ (where $\cdot$ is the iner product in $\mathbb{R}^d$), is $L^\infty\left(\mathbb{R}^d, dx\right)$-unique with respect to the topology $\mathcal{C}\left(L^\infty, L^1\right)$. As consequence, is obtained the $L^1\left(\mathbb{R}^d, dx\right)$-uniqueness of weak solution for the Fokker-Planck equation associated with $\mathcal{A}^V$. This result was reported in the conference EQUADIFF2007 held on August 2007 at Vienna.

2 Uniqueness of pre-generators on the dual of a Banach space

One of the main results of this paper concern the uniqueness of pre-generators on the dual of a Banach space. Recall that a linear operator $\mathcal{A} : \mathcal{D} \longrightarrow \mathcal{X}^*$ with the domain $\mathcal{D}$ dense in $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$ is said to be *a pre-generator* in $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$, if there exists some C_0-semigroup on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$ such that its generator $\mathcal{L}$ extends $\mathcal{A}$.
The main results of this section is

THEOREM 2.1. *Let $\mathcal{A} : \mathcal{D} \longrightarrow \mathcal{X}^*$ be a linear operator with domain $\mathcal{D}$ dense in $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$. Suppose that there exists a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$ such that its generator $\mathcal{L}$ extends $\mathcal{A}$ (i.e. $\mathcal{A}$ is a pre-generator).*
If $\mathcal{D}$ is not a core of $\mathcal{L}$, then there exists an infinite number of extensions of $\mathcal{A}$ which are generators.

For the proof of Theorem 2.1 we need to use some perturbation result. Perturbation theory has long been a very useful tool in the hand of the analyst and physicist. A very elegant brief introduction to one-parameter semigroups is given in the treatise of Kato [16] where on can find all results on perturbation theory. The perturbation by bounded operators is due to Phillips [24] who also investigate permanence of smoothness properties by this kind of perturbation. The perturbation by continuous operators on the graph norm of the generator is due to Desch and Schappacher [7].
Next lemma (comunicated by professor Liming Wu), which presents a Desch-Schappacher perturbation result for C_0-semigroups on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$, play a key rolle in the proof of Theorem 2.1:

LEMMA 2.2. *Let $(\mathcal{X}, \|\,.\,\|)$ be a Banach space, $\mathcal{L}$ the generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$ and C a linear operator on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$ with domain $\mathcal{D}(C) \supset \mathcal{D}(\mathcal{L})$.*

(i) If C is $\mathcal{C}(\mathcal{X}^*, \mathcal{X})$-continuous, then $\mathcal{L} + C$ with domain $\mathcal{D}(\mathcal{L} + C) = \mathcal{D}(\mathcal{L})$ is the generator of some C_0-semigroup on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$.

(ii) If $C : \mathcal{D}(\mathcal{L}) \to \mathcal{D}(\mathcal{L})$ is continuous with respect to the graph topology of $\mathcal{L}$ induced by the topology $\mathcal{C}(\mathcal{X}^*, \mathcal{X})$, then $\mathcal{L} + C$ with domain $\mathcal{D}(\mathcal{L} + C) = \mathcal{D}(\mathcal{L})$ is the generator of some C_0-semigroup on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$.

Proof. (i) By the [30, Theorem 1.4, p.564] and using Lemma 1.1, $\mathcal{L}^*$ is the generator of the C_0-semigroup $\{T^*(t)\}_{t\geq 0}$ on $(\mathcal{X}, \mathcal{C}(\mathcal{X}, \mathcal{X}^*_{\mathcal{C}})) = (\mathcal{X}, \|\,.\,\|)$. Under the condition on C, by [30, Lemma 1.12, p.568] it follows that the operator C^* is bounded on $(\mathcal{X}, \|\,.\,\|)$. By a well known perturbation result (see [6, Theorem 1, p.68]), we find that $\mathcal{L}^* + C^* = (\mathcal{L} + C)^*$ is the generator of some C_0-semigroup on $(\mathcal{X}, \|\,.\,\|)$. By using again [30, Theorem 1.4, p.564], we obtain that $(\mathcal{L} + C)^{**}$ is the generator of some C_0-semigroup on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$. Moreover, $\mathcal{D}((\mathcal{L} + C)^*)$ is dense in $(\mathcal{X}, \|\,.\,\|)$. Hence $\mathcal{D}((\mathcal{L} + C)^*)$ is dense in $(\mathcal{X}, \sigma(\mathcal{X}, \mathcal{X}^*))$. Then by [26, Theorem 7.1, p.155] it follows that

$$(\mathcal{L} + C)^{**} = \overline{(\mathcal{L} + C)}^{\sigma(\mathcal{X}^*, \mathcal{X})}$$

Since C is $\mathcal{C}(\mathcal{X}^*, \mathcal{X})$-continuous, by [30, Lemma 1.5, p.564] it follows that C is $\sigma(\mathcal{X}^*, \mathcal{X})$-continuous hence $\sigma(\mathcal{X}^*, \mathcal{X})$-closed. Consequently

$$\mathcal{L} + C = \overline{(\mathcal{L} + C)}^{\sigma(\mathcal{X}^*, \mathcal{X})}$$

from where it follows that $(\mathcal{L} + C)^{**} = \mathcal{L} + C$. Hence $\mathcal{L} + C$ is the generator of some C_0-semigroup on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$.

(ii) We will follows closely the proof of Arendt [2, Theorem 1.31, p.45]. Remark that

$C : \mathcal{D}(\mathcal{L}) \to \mathcal{D}(\mathcal{L})$ is continuous with respect to the graph topology of $\mathcal{L}$ induced by the topology $\mathcal{C}(\mathcal{X}^*, \mathcal{X})$ if and only if for all $\lambda > \omega_0$ (where ω_0 is the real constatnte in the definition of the C_0-semigroup $\{T(t)\}_{t\geq 0}$) the operator

$$\tilde{C} := (\lambda I - \mathcal{L})CR(\lambda; \mathcal{L})$$

is continuous on $\mathcal{X}^*$ with respect to the topology $\mathcal{C}(\mathcal{X}^*, \mathcal{X})$. Consequently, by (i) we find that $\mathcal{L} + \tilde{C}$ is the generator of some C_0-semigroup on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$. We shall prove that $\mathcal{L} + \tilde{C}$ is similar to $\mathcal{L} + C$. Remark that C is continuous with respect to the graph norm $\| \, . \, \|^* + \|\mathcal{L}. \, \|^*$. By the prove of [2, Theorem 1.31, p.45], there exists some $\lambda > \omega_0$ such that the operators

$$U := I - CR(\lambda; \mathcal{L}) \quad \text{and} \quad U^{-1}$$

are bounded on $(\mathcal{X}^*, \| \, . \, \|^*)$. Moreover

$$U(\mathcal{L} + \tilde{C})U^{-1} = U(\mathcal{L} - \lambda I + \tilde{C})U^{-1} + \lambda I =$$

$$= U[\mathcal{L} - \lambda I + (\lambda I - \mathcal{L})CR(\lambda; \mathcal{L})]U^{-1} + \lambda I =$$

$$= U(\mathcal{L} - \lambda I)[I - CR(\lambda; \mathcal{L})]U^{-1} + \lambda I =$$

$$= U(\mathcal{L} - \lambda I) + \lambda I = [I - CR(\lambda; \mathcal{L})](\mathcal{L} - \lambda I) + \lambda I =$$

$$= \mathcal{L} - \lambda I + C + \lambda I = \mathcal{L} + C$$

Now we have only to prove that U and U^{-1} are continuous with respect to the topology $\mathcal{C}(\mathcal{X}^*, \mathcal{X})$. Since $CR(\lambda; \mathcal{L}) = R(\lambda; \mathcal{L})\tilde{C}$ is continuous with respect to the topology $\mathcal{C}(\mathcal{X}^*, \mathcal{X})$, it follows that $U = I - CR(\lambda; \mathcal{L})$ is continuous with respect to the topology $\mathcal{C}(\mathcal{X}^*, \mathcal{X})$. On the other hand, by [30, Lemma 1.5, p.564], U^* and

$[CR(\lambda;\mathcal{L})]^*$ are continuous on $(\mathcal{X}, \| \, . \, \|)$. By Phillips theorem [17, Proposition 5.9, p.246], $1 \in \rho([CR(\lambda;\mathcal{L})]^*)$ if and only if $1 \in [CR(\lambda;\mathcal{L})]^{**}$ and

$$[I - ([CR(\lambda;\mathcal{L})]^*)^{-1}]^* = (I - [CR(\lambda;\mathcal{L})]^{**})^{-1}$$

But by [26, Theorem 1.1, p.155] we have $[CR(\lambda;\mathcal{L})]^{**} = CR(\lambda;\mathcal{L})$ and the right hand side above becomes U^{-1}. Hence U^{-1}, being the dual of some bounded operator on $(\mathcal{X}, \| \, . \, \|)$, is continuous on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$ by [30, Lemma 1.5, p.564] and the proof of lemma is completed. $\Box$

Now we are able to give

Proof of Theorem 2.1. We will follows closely the proof of Arendt [2, Theorem 1.33, p.46]. Endow $\mathcal{D}(\mathcal{L})$ with the graph topology $\mathcal{C}_{\mathcal{L}}(\mathcal{X}^*, \mathcal{X})$ of $\mathcal{L}$ induced by the topology $\mathcal{C}(\mathcal{X}^*, \mathcal{X})$. If in contrary $\mathcal{D}$ is not a core of $\mathcal{L}$, then $\mathcal{D}$ is not dense in $\mathcal{D}(\mathcal{L})$ with respect to the graph topology $\mathcal{C}_{\mathcal{L}}(\mathcal{X}^*, \mathcal{X})$ of $\mathcal{L}$. By Hahn-Banach theorem there exist some non-zero linear functional ϕ continuous on $\mathcal{D}(\mathcal{L})$ with respect to the graph topology $\mathcal{C}_{\mathcal{L}}(\mathcal{X}^*, \mathcal{X})$ of $\mathcal{L}$ such that $\phi(x) = 0$ for all $x \in \mathcal{D}$. Fix some $u \in \mathcal{D}(\mathcal{L})$, $u \neq 0$, and consider the linear operator

$$C : \mathcal{D}(\mathcal{L}) \longrightarrow \mathcal{D}(\mathcal{L})$$

$$Cx = \phi(x)u \quad , \quad \forall x \in \mathcal{D}(\mathcal{L}).$$

Then C is continuous with respect to the graph topology $\mathcal{C}_{\mathcal{L}}(\mathcal{X}^*, \mathcal{X})$ of $\mathcal{L}$ on $\mathcal{D}(\mathcal{L})$. By (Desch-Schappacher perturbation) Lemma 2.2 it follows that $\mathcal{L} + C$ is the generator of some C_0-semigroupe on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$ and

$$(\mathcal{L} + C)/_{\mathcal{D}} = \mathcal{L}/_{\mathcal{D}} = \mathcal{A}$$

It is obvious that an infinite number of generators can be constructed in that way. $\Box$

3 $L^\infty(\mathbb{R}^d, dx)$-uniqueness of generalized Schrödinger operators

In this section we consider the generalized Schrödinger operator

$$\mathcal{A}^V f := \frac{1}{2}\Delta f + b \cdot \nabla f - V f \quad , \quad \forall f \in C_0^\infty(\mathbb{R}^d)$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ is a measurable locally bounded vector field and $V : \mathbb{R}^d \to \mathbb{R}$ is a locally bounded potential. The study of this operator has attracted much attention both from the people working on Nelson's stochastic mechanics (Carmona [4], Meyer and Zheng [22], etc.) and from those working on the theory of Dirichlet forms (Albeverio, Brasche and Röckner [1]). In the case where $V = 0$, the essential self-adjointness of $\mathcal{A} := \frac{1}{2}\Delta + b \cdot \nabla$ in L^2 has been completely charaterized in the works of Wielens [27] and Liskevitch [21]. L^1-uniqueness of this operator has been introduced and studied by Wu [29], its L^p-uniqueness has been studied by Eberle [9] for $p \in [1, \infty)$ and by Wu and Zhang [30] for $p = \infty$.

In accord with the Theorem 2.1, we can introduce $L^\infty\left(\mathbb{R}^d, dx\right)$-uniqueness of pre-generators in a very natural form:

DEFINITION 3.1. We say that a pre-generator $\mathcal{A}$ is $\left(L^\infty\left(\mathbb{R}^d, dx\right), \mathcal{C}\left(L^\infty, L^1\right)\right)$-unique, if there exists only one C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $\left(L^\infty\left(\mathbb{R}^d, dx\right), \mathcal{C}\left(L^\infty, L^1\right)\right)$ such that its generator $\mathcal{L}$ is an extension of $\mathcal{A}$.

This uniqueness notion has been used by Arendt [2], Röckner [25], Wu [28] and [29], Eberle [9], Arendt, Metafune and Pallara [3], Wu and Zhang [30], Lemle [18] and others in different contexts. The next characterisation of $\left(L^\infty\left(\mathbb{R}^d, dx\right), \mathcal{C}\left(L^\infty, L^1\right)\right)$-uniqueness of pre-generators is wery useful in applications (for others characterisations of the uniqueness of pre-generators we strongly recommanded for the reader the excelent article of WU and ZHANG [30]):

THEOREM 3.2. *Let $\mathcal{A}$ be a linear operator on $\left(L^\infty\left(\mathbb{R}^d, dx\right), \mathcal{C}\left(L^\infty, L^1\right)\right)$ with domain $\mathcal{D}$ (the test-function space) which is assumed to be dense in $\left(L^\infty\left(\mathbb{R}^d, dx\right), \mathcal{C}\left(L^\infty, L^1\right)\right)$. Assume that there is a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $\left(L^\infty\left(\mathbb{R}^d, dx\right), \mathcal{C}\left(L^\infty, L^1\right)\right)$ such that its generator $\mathcal{L}$ is an extension of $\mathcal{A}$ (i.e., $\mathcal{A}$ is a pre-generator). The following assertions are equivalents:*
(i) $\mathcal{A}$ is $\left(L^\infty\left(\mathbb{R}^d, dx\right), \mathcal{C}\left(L^\infty, L^1\right)\right)$-unique;
(ii) $\mathcal{D}$ is a core of $\mathcal{L}$;
(iii) for some $\lambda > \omega_0$ (where $\omega_0 \in \mathbb{R}$ is the constant in the definition of C_0-semigroup $\{T(t)\}_{t\geq 0}$), the range $(\lambda I - \mathcal{A})(\mathcal{D})$ is dense in $\left(L^\infty\left(\mathbb{R}^d, dx\right), \mathcal{C}\left(L^\infty, L^1\right)\right)$;
(iv) (Liouville property) for some $\lambda > \omega_0$, if $h \in \mathcal{D}(\mathcal{A}^*)$ satisfies $(\lambda I - \mathcal{A}^*)h = 0$, then $h = 0$;
(v) (uniqueness of weak solutions for the dual Cauchy problem) for every $f \in \left(L^1\left(\mathbb{R}^d, dx\right), \| \,.\, \|_1\right)$, the dual Cauchy problem

$$\begin{cases} \partial_t u(t,x) = \mathcal{A}^* u(t,x) \\ u(0,x) = f(x) \end{cases}$$

has a $\left(L^1\left(\mathbb{R}^d, dx\right), \| \,.\, \|_1\right)$-unique weak solution $u(t,x) = T^*(t)f(x)$.

Our main purpose in this section is to find some sufficient condition to assure the $L^\infty(\mathbb{R}^d, dx)$-uniqueness of $(\mathcal{A}^V, C_0^\infty(\mathbb{R}^d))$ with respect to the topology $\mathcal{C}\left(L^\infty, L^1\right)$ in the case where $V \geq 0$.

At first, we must remark that the generalized Schrödinger operator $(\mathcal{A}^V, C_0^\infty(\mathbb{R}^d))$ is a pre-generator on $\left(L^\infty(\mathbb{R}^d, dx), \mathcal{C}\left(L^\infty, L^1\right)\right)$. Indeed, if we consider the Feynman-Kac semigroup $\left\{P_t^V\right\}_{t\geq 0}$given by

$$P_t^V f(x) := \mathbb{E}^x 1_{[t<\tau_e]} f(X_t) e^{-\int\limits_0^t V(X_s)\, ds}$$

where $(X_t)_{0\leq t<\tau_e}$ is the diffusion generated by $\mathcal{A}$ and τ_e is the explosion time, then by [30, Theorem 1.4] $\left\{P_t^V\right\}_{t\geq 0}$is a C_0-semigroup on $L^\infty(\mathbb{R}^d, dx)$ with respect to the topology $\mathcal{C}\left(L^\infty, L^1\right)$. Let ∂ be the point at infinity of $\mathbb{R}^d$. If we put $X_t = \partial$ after the explosion time $t \geq \tau_e$, then by Ito's formula it follows for any $f \in C_0^\infty(\mathbb{R}^d)$ that

$$f(X_t) - f(x) - \int\limits_0^t \mathcal{A}^V f(X_s)\, ds$$

is a local martingale. As it is bounded over bounded times intervals, it is a true martingale. Thus by taking the expectation under $\mathbb{P}_x$, we get

$$P_t^V f(x) - f(x) = \int\limits_0^t P_s^V \mathcal{A}^V f(x)\, ds \quad , \quad \forall t \geq 0.$$

Therefore f belongs to the domain of the generator $\mathcal{L}_{(\infty)}^V$ of C_0-semigroup $\left\{P_t^V\right\}_{t\geq 0}$on $(L^\infty(\mathbb{R}^d, dx), \mathcal{C}\left(L^\infty, L^1\right))$. Consequently, $(\mathcal{A}^V, C_0^\infty(\mathbb{R}^d))$ is a pre-generator on $L^\infty(\mathbb{R}^d, dx)$ with respect to the topology $\mathcal{C}\left(L^\infty, L^1\right)$ and we can apply the Theorem 3.2 to study the $(L^\infty(\mathbb{R}^d, dx), \mathcal{C}\left(L^\infty, L^1\right))$-uniqueness of this operator.

3.1 The one-dimensional case

The purpose of this subsection is to study the L^∞-uniqueness of one-dimensional operator

$$\mathcal{A}_1^V f = a(x) f^{''} + b(x) f^{'} - V(x) f \quad , \quad f \in C_0^\infty(x_0, y_0)$$

where $-\infty \leq x_0 < y_0 \leq \infty$ and the coefficients a, b and V satisfy the next properties

$$a(x),\ b(x) \in L_{loc}^\infty(x_0, y_0; dx)$$

$$V(x) \in L^{\infty}_{loc}(x_0, y_0; dx), \;\; V(x) \geq 0$$

and the following very weak ellipticity condition

$$a(x) > 0 \quad dx - \text{a.e.}$$

$$\frac{1}{a(x)}, \quad \frac{b(x)}{a(x)} \in L^{1}_{loc}(x_0, y_0; dx)$$

where $L^{\infty}_{loc}(x_0, y_0; dx)$, respectively $L^{1}_{loc}(x_0, y_0; dx)$, denotes the space of real Lebesgue measurable functions which are essentially bounded, respectively integrable, with respect to Lebesgue measure on any compact sub-interval of (x_0, y_0).

Fix a point $c \in (x_0, y_0)$ and let

$$\rho(x) = \frac{1}{a(x)} e^{\int\limits_c^x \frac{b(t)}{a(t)}\, dt} \quad .$$

be *the speed measure of Feller* and let

$$\alpha(x) = e^{\int\limits_c^x \frac{b(t)}{a(t)}\, dt}$$

be *the scale function of Feller.* It is easy to see that

$$\left\langle \mathcal{A}_1^V f, g \right\rangle_\rho = \left\langle f, \mathcal{A}_1^V g \right\rangle_\rho \quad , \quad \forall f, g \in C_0^{\infty}(x_0, y_0)$$

where

$$\langle f, g \rangle_\rho = \int\limits_{x_0}^{y_0} f(x) g(x) \rho(x)\, dx \quad .$$

For $f \in C_0^{\infty}(x_0, y_0)$, we can write $\mathcal{A}_1^V$ in the Feller form:

$$\mathcal{A}_1^V = a(x) f'' + b(x) f' - V(x) f = \frac{\alpha(x)}{\rho(x)} f'' + \frac{a(x)\alpha'(x)}{\alpha(x)} f' - V(x) f =$$

$$= \frac{\alpha(x)}{\rho(x)} f'' + \frac{\alpha'(x)}{\rho(x)} f' - V(x) f = \frac{1}{\rho(x)} \left[\alpha(x) f'\right]' - V(x) f$$

and the assumptions concerning the coeficients $a(x)$ and $b(x)$ can be writen as

- $\rho(x) > 0$, dx-a.e. and $\rho \in L^1_{loc}(x_0, y_0; dx)$
- $\alpha(x) > 0$ everywhere and α is absolutely continuous
- $\alpha/\rho, \quad \alpha^{'}/\rho \in L^{\infty}_{loc}(x_0, y_0; dx)$.

Now consider the operator $(\mathcal{A}_1^V, C_0^{\infty}(x_0, y_0))$ as an operator on $L^{\infty}(x_0, y_0; \rho dx)$ which is endowed with the topology $\mathcal{C}(L^{\infty}(x_0, y_0, \rho dx), L^1(x_0, y_0, \rho dx))$. We begin with a series of lemmas.

LEMMA 3.3. *Let $(\mathcal{A}_1^V)^* : \mathcal{D}((\mathcal{A}_1^V)^*) \subset L^1(x_0, y_0; \rho dx) \to L^1(x_0, y_0; \rho dx)$ be the adjoint operator of $\mathcal{A}_1^V$. Let $\lambda > 0$ and let $u \in L^1(x_0, y_0; \rho dx)$ be in $\mathcal{D}((\mathcal{A}_1^V)^*)$ such that*

$$(\mathcal{A}_1^V)^* u = \lambda u.$$

Then u solves the ordinary differential equation

$$\left(\alpha u^{'}\right)^{'} = \lambda u \rho + V u \rho$$

in the following sense: u has an absolutely continuous dx-version $\hat{u}$ such that $\hat{u}^{'}$ is absolutely continuous and

$$\left(\alpha \hat{u}^{'}\right)^{'} = \lambda \hat{u} \rho + V \hat{u} \rho.$$

Proof. The sufficiency follows easily by integration by parts.

Below we prove the necessity. Let $x_0 < x_1 < y_1 < y_0$. The space of distributions on (x_1, y_1) is denoted by $\mathcal{D}'(x_1, y_1)$.

(I) We recall that if $k \geq 1$ and $T_1, T_2 \in \mathcal{D}'(x_1, y_1)$ satisfy $T_1^{(k)} = T_2^{(k)}$ i.e.

$$\int_{x_1}^{y_1} T_1 f^{(k)}(x)\, dx = \int_{x_1}^{y_1} T_2 f^{(k)}(x)\, dx$$

for any $f \in C_0^\infty(x_1, y_1)$, then there exists a polynomial w such that $T_1 = T_2 + w$.

(II) Let $u \in L^1(x_0, y_0; \rho dx)$ be in $\mathcal{D}((\mathcal{A}_1^V)^*)$ such that

$$(\mathcal{A}_1^V)^* u = \lambda u.$$

Then for $f \in C_0^\infty(x_1, y_1)$ we have:

$$\begin{aligned}
&\int_{x_1}^{y_1} u \left(\alpha f'\right)' \, dx = \int_{x_1}^{y_1} u \mathcal{A}_1^V f \rho \, dx + \int_{x_1}^{y_1} u V f \rho \, dx = \\
= \quad & \left\langle u, \mathcal{A}_1^V f \right\rangle_\rho + \langle u, V f \rangle_\rho = \left\langle (\mathcal{A}_1^V)^* u, f \right\rangle_\rho + \langle u, V f \rangle_\rho = \\
= \quad & \langle \lambda u, f \rangle_\rho + \langle u, V f \rangle_\rho = \lambda \int_{x_1}^{y_1} u f \rho \, dx + \int_{x_1}^{y_1} u V f \rho \, dx.
\end{aligned}$$

From

$$|f(x)| = \left| \int_{x_1}^{x} f'(t) \, dt \right| \le \int_{x_1}^{x} |f'(t)| \, dt \le \int_{x_1}^{y_1} |f'(t)| \, dt$$

it follows that

$$\|f\|_{L^\infty(x_1, y_1; dx)} \le \|f'\|_{L^1(x_1, y_1; dx)}$$

and we have

$$\begin{aligned}
&\left| \int_{x_1}^{y_1} u \left[\alpha f'' + \alpha' f' \right] \, dx \right| = \left| \int_{x_1}^{y_1} u \left(\alpha f' \right)' \, dx \right| \le \\
\le \quad & \lambda \left| \int_{x_1}^{y_1} u f \rho \, dx \right| + \left| \int_{x_1}^{y_1} u V f \rho \, dx \right| \le \\
\le \quad & \left[\lambda \left\| u \rho \right\|_{L^1(x_0, y_0; dx)} + \left\| u V \rho \right\|_{L^1(x_1, y_1; dx)} \right] \|f\|_{L^\infty(x_1, y_1; dx)} \le \\
\le \quad & C \left\| f' \right\|_{L^1(x_1, y_1; dx)}
\end{aligned}$$

where

$$C = \lambda \left\| u \rho \right\|_{L^1(x_0, y_0; dx)} + \left\| u V \rho \right\|_{L^1(x_1, y_1; dx)}$$

is independent of f. The above inequality means that the linear functional

$$l_u(\eta) := \int_{x_1}^{y_1} u\left(\alpha\eta' + \alpha'\eta\right)\, dx$$

where $\eta \in \left\{ f' \mid f \in C_0^\infty(x_1, y_1) \right\} \subset L^1(x_1, y_1; dx)$, is continuous with respect to the $L^1(x_1, y_1; dx)$-norm. Thus by the Hahn-Banach's theorem and the fact that the dual of $L^1(x_1, y_1; dx)$ is $L^\infty(x_1, y_1; dx)$, there exists $v \in L^\infty(x_1, y_1; dx)$ such that

$$l_u(\eta) := \int_{x_1}^{y_1} u\left(\alpha\eta' + \alpha'\eta\right)\, dx = \int_{x_1}^{y_1} v\eta\, dx$$

which implies

$$\int_{x_1}^{y_1} u\alpha\eta'\, dx = \int_{x_1}^{y_1} \left(v - u\alpha'\right)\eta\, dx = \int_{x_1}^{y_1} h\eta'\, dx$$

where

$$h(x) = -\int_{x_1}^{x} \left[v(t) - u(t)\alpha'(t)\right]\, dt$$

is an absolutely continuous function on (x_1, y_1). It follows from **(I)** that there exists a polynomial w such that

$$u\alpha = h + w$$

on (x_1, y_1) in the sense of distributions, hence $u\alpha = h + w$ a.e. on (x_1, y_1).

(III) Since $\alpha > 0$ is absolutely continuous, the equality

$$u = \alpha^{-1}(h + w) \quad \text{a.e.}$$

shows that u also has an absolutely continuous version

$$\tilde{u} := \alpha^{-1}(h + w).$$

(IV) Now we have

$$\lambda \int_{x_1}^{y_1} \tilde{u} f \rho \, dx = \int_{x_1}^{y_1} \tilde{u} \left(\alpha f' \right)' dx - \int_{x_1}^{y_1} \tilde{u} V f \rho \, dx =$$
$$= - \int_{x_1}^{y_1} \tilde{u}' \alpha f' \, dx - \int_{x_1}^{y_1} \tilde{u} V f \rho \, dx.$$

so that

$$\int_{x_1}^{y_1} (\lambda \tilde{u} \rho + \tilde{u} V \rho) \, dx = - \int_{x_1}^{y_1} \tilde{u}' \alpha f' \, dx.$$

Hence

$$\left(\alpha \tilde{u}' \right)' = \lambda \tilde{u} \rho + \tilde{u} V \rho \in L^1(x_1, y_1; dx)$$

in the sense of distributions. Then $\alpha \tilde{u}'$ has an absolutely continuous version, so is $\tilde{\tilde{u}}'$ (a primitive of $\lambda \tilde{u} \rho + \tilde{u} V \rho$) on (x_1, y_1) and

$$\tilde{\tilde{u}}' = \lambda \tilde{u} \rho + \tilde{u} V \rho \quad \text{a.e.}$$

(V) From the above discution we have

$$\alpha \tilde{u}' = \tilde{\tilde{u}} \quad \text{a.e.}$$

which implies that

$$\tilde{u}' = \alpha^{-1} \tilde{\tilde{u}} \quad \text{a.e.}$$

Since $\alpha^{-1} \tilde{\tilde{u}}$ is absolutely continuous, we get that $\tilde{u}$, hence u has a version $\hat{u}$ (a primitive of $\alpha^{-1} \tilde{\tilde{u}}$) such that

$$\hat{u}' = \alpha^{-1} \tilde{\tilde{u}}$$

is absolutely continuous. We then go back to **(IV)**, using $\hat{u}$ in place of $\tilde{u}$, to obtain

$$\left(\alpha \hat{u}' \right)' = \lambda \hat{u} \rho + V \hat{u} \rho.$$

The lemma is thus proved since (x_1, y_1) is an arbitrary relatively compact subinterval of (x_0, y_0). $\square$

LEMMA 3.4. *Let $\lambda > 0$ and let $u \in L^1(x_0, y_0; \rho dx)$ be such that*

$$(\mathcal{A}_1^V)^* u = \lambda u$$

in the sense of Lemma 3.3. We may suppose that u is an absolutely continuous version such that $u^{'}$ is absolutely continuous. Let $c_1 \in (x_0, y_0)$ such that $u(c_1) > 0$.
(i) if $u^{'}(c_1) > 0$, then $u^{'}(y) > 0$ for all $y \in (c_1, y_0)$;
(ii) if $u^{'}(c_1) < 0$, then $u^{'}(x) < 0$ for all $x \in (x_0, c_1)$.

Proof. (i) Suppose $u^{'}(c_1) > 0$. Let

$$\hat{y} = \sup \left\{ y \geq c_1 \;\middle|\; u^{'}(z) > 0, \;\; \forall z \in [c_1, y) \right\} \quad .$$

It is clear that $\hat{y} > c_1$ and

$$u(t) \geq u(c_1) > 0 \quad , \quad \forall t \in [c_1, \hat{y}].$$

From the hypothesis

$$(\mathcal{A}_1^V)^* u = \lambda u$$

it follows that

$$\left(\alpha u^{'}\right)^{'} = \lambda u \rho + u V \rho.$$

Then for any $y \in (c_1, y_0)$ we have

$$\alpha(y) u^{'}(y) - \alpha(c_1) u^{'}(c_1) = \int_{c_1}^{y} \rho(t)[\lambda + V(t)] u(t) \, dt \quad .$$

If $\hat{y} < y_0$, then

$$\alpha(\hat{y})u^{'}(\hat{y}) - \alpha(c_1)u^{'}(c_1) = \int\limits_{c_1}^{\hat{y}} \rho(t)[\lambda + V(t)]u(t)\, dt$$

from where it follows that

$$\alpha(\hat{y})u^{'}(\hat{y}) = \alpha(c_1)u^{'}(c_1) + \int\limits_{c_1}^{\hat{y}} \rho(t)[\lambda + V(t)]u(t)\, dt > \alpha(c_1)u^{'}(c_1) > 0.$$

Then $u^{'}(\hat{y}) > 0$. Hence $u^{'}(t) > 0$ for all $t \in [\hat{y}, \hat{y} + \varepsilon]$ for small $\varepsilon > 0$, which contradicts the definition of $\hat{y}$.

(ii) In the same way on can prove that if $u^{'}(c_1) < 0$, then $u^{'}(x) < 0$, for all $x \in (x_0, c_1)$.

□

LEMMA 3.5. *There exists two strictely positive functions u_k, $k = 1, 2$ on (x_0, y_0) such that*

(i) for $k = 1, 2$, $u_k^{'}$ is absolutely continuous and

$$\left(\alpha u_k^{'}\right)^{'} = \lambda u_k \rho + u_k V \rho \quad a.e.$$

where $\lambda > 0$;

(ii) $u_1^{'} > 0$ and $u_2^{'} < 0$ over (x_0, y_0).

Proof. The function u_2 was constructed by Feller [10, Lemma 1.9] in the case where $a = 1$ and $V = 0$, but his prove works in the actual general framework. □

The main result of this subsection is

THEOREM 3.6. *The one-dimensional operator $(\mathcal{A}_1^V, C_0^\infty(x_0, y_0))$ is $L^\infty(x_0, y_0; \rho dx)$-unique with respect to the topology $\mathcal{C}(L^\infty(x_0, y_0; \rho dx), L^1(x_0, y_0; \rho dx))$ if an only if both*

$$(*) \qquad \int\limits_{c}^{y_0} \rho(y) \sum_{n=0}^{\infty} \phi_n(y)\, dy = +\infty$$

and

$$(**) \qquad \int_{x_0}^{c} \rho(x) \sum_{n=0}^{\infty} \psi_n(x)\, dx = +\infty$$

hold, where $c \in (x_0, y_0)$*,* $\lambda > 0$ *and*

$$\phi_n(y) = \int_c^y \frac{1}{\alpha(r_n)}\, dr_n \int_c^{r_n} \rho(t_n)[\lambda + V(t_n)]\phi_{n-1}(t_n)\, dt_n, \quad n \geq 1, \quad \phi_0(y) = 1$$

and

$$\psi_n(x) = \int_x^c \frac{1}{\alpha(r_n)}\, dr_n \int_{r_n}^{c} \rho(t_n)[\lambda + V(t_n)]\psi_{n-1}(t_n)\, dt_n, \quad n \geq 1, \quad \psi_0(x) = 1.$$

Proof. $\Rightarrow$ Let $(\mathcal{A}_1^V, C_0^\infty(x_0, y_0))$ be $L^\infty(x_0, y_0; \rho dx)$-unique with respect to the topology $\mathcal{C}(L^\infty(x_0, y_0; \rho dx), L^1(x_0, y_0; \rho dx))$ and assume that (**) (similar in the case (*)) doesn't hold, that is

$$\int_{x_0}^{c} \rho(x) \sum_{n=0}^{\infty} \psi_n(x)\, dx < +\infty$$

where $c \in (x_0, y_0)$ is fixed and $\lambda > 0$. We prove that there exists $u \in L^1(x_0, y_0; \rho dx)$, $u \neq 0$ such that

$$\left[\lambda I - (\mathcal{A}_1^V)^*\right] u = 0 \quad \textit{in the sense of distributions}$$

which is in contradiction with the $L^\infty(x_0, y_0; \rho dx)$-uniqueness of $(\mathcal{A}_1^V, C_0^\infty(x_0, y_0))$. Indeed, by Lemma 3.5 there exists a function u strictely positive on (x_0, y_0) such that u' is absolutely continuous, $u' < 0$ over (x_0, y_0) and

$$\left(\alpha u'\right)' = \rho(\lambda + V)u.$$

Below we shall prove that $u \in L^1(x_0, y_0; \rho dx)$.

(**I**) *integrability near* y_0

For $y \in (c, y_0)$ we have

$$\alpha(y)u^{'}(y) - \alpha(c)u^{'}(c) = \int_c^y \rho(t)[\lambda + V(t)]u(t)\,dt.$$

Then

$$0 \geq \alpha(y)u^{'}(y) = \alpha(c)u^{'}(c) + \int_c^y \rho(t)[\lambda + V(t)]u(t)\,dt$$

which implies that

$$\int_c^y u(t)\rho(t)\,dt \leq \int_c^y \rho(t)[\lambda + V(t)]u(t)\,dt \leq -\alpha(c)u^{'}(c) < +\infty.$$

(**II**) *integrability near* x_0

For $x \in (x_0, c)$ we have

$$\alpha(c)u^{'}(c) - \alpha(x)u^{'}(x) = \int_x^c \rho(t)[\lambda + V(t)]u(t)\,dt$$

so that

$$\alpha(x)u^{'}(x) = \alpha(c)u^{'}(c) - \int_x^c \rho(t)[\lambda + V(t)]u(t)\,dt.$$

Moreover for $c_0 \in (x, c)$ we have:

$$u(x) = u(c) - \int_x^c u^{'}(r)\,dr =$$

$$= u(c) - \int_x^c \left\{ \frac{\alpha(c)u^{'}(c)}{\alpha(r)} - \frac{1}{\alpha(r)} \int_r^c \rho(t)[\lambda + V(t)]u(t)\,dt \right\} dr =$$

$$= u(c) - \alpha(c)u^{'}(c) \int_x^c \frac{1}{\alpha(r)}\,dr + \int_x^c \frac{1}{\alpha(r)}\,dr \int_r^c \rho(t)[\lambda + V(t)]u(t)\,dt =$$

$$= u(c) - \alpha(c)u^{'}(c)\left[\int\limits_x^{c_0} \frac{1}{\alpha(r)}\,dr + \int\limits_{c_0}^{c} \frac{1}{\alpha(r)}\,dr\right] +$$

$$+ \int\limits_x^c \frac{1}{\alpha(r)}\,dr \int\limits_r^c \rho(t)[\lambda + V(t)]u(t)\,dt =$$

$$= u(c) - \alpha(c)u^{'}(c)\int\limits_x^{c_0} \frac{1}{\alpha(r)} \cdot \frac{\int\limits_{c_0}^{c}\rho(t)[\lambda + V(t)]\,dt}{\int\limits_{c_0}^{c}\rho(t)[\lambda + V(t)]\,dt}\,dr -$$

$$-\alpha(c)u^{'}(c)\int\limits_{c_0}^{c} \frac{1}{\alpha(r)}\,dr + \int\limits_x^c \frac{1}{\alpha(r)}\,dr \int\limits_r^c \rho(t)[\lambda + V(t)]u(t)\,dt =$$

$$= u(c) - \frac{\alpha(c)u^{'}(c)}{\int\limits_{c_0}^{c}\rho(t)[\lambda + V(t)]\,dt}\int\limits_x^{c_0} \frac{1}{\alpha(r)}\,dr \int\limits_{c_0}^{c}\rho(t)[\lambda + V(t)]\,dt -$$

$$-\alpha(c)u^{'}(c)\int\limits_{c_0}^{c} \frac{1}{\alpha(r)}\,dr + \int\limits_x^c \frac{1}{\alpha(r)}\,dr \int\limits_r^c \rho(t)[\lambda + V(t)]u(t)\,dt \le$$

$$\le u(c) - \frac{\alpha(c)u^{'}(c)}{\int\limits_{c_0}^{c}\rho(t)[\lambda + V(t)]\,dt}\int\limits_x^{c_0} \frac{1}{\alpha(r)}\,dr \int\limits_{r}^{c}\rho(t)[\lambda + V(t)]\,dt -$$

$$-\alpha(c)u^{'}(c)\int\limits_{c_0}^{c} \frac{1}{\alpha(r)}\,dr + \int\limits_x^c \frac{1}{\alpha(r)}\,dr \int\limits_r^c \rho(t)[\lambda + V(t)]u(t)\,dt \le$$

$$\le u(c) - \frac{\alpha(c)u^{'}(c)}{\int\limits_{c_0}^{c}\rho(t)[\lambda + V(t)]\,dt}\int\limits_x^{c} \frac{1}{\alpha(r)}\,dr \int\limits_{r}^{c}\rho(t)[\lambda + V(t)]\,dt -$$

$$-\alpha(c)u^{'}(c)\int\limits_{c_0}^{c} \frac{1}{\alpha(r)}\,dr + \int\limits_x^c \frac{1}{\alpha(r)}\,dr \int\limits_r^c \rho(t)[\lambda + V(t)]u(t)\,dt.$$

Thus:

$$u(x) \leq u(c) - \alpha(c)u^{'}(c) \int\limits_{c_0}^{c} \frac{1}{\alpha(r)}\, dr -$$

$$- \frac{\alpha(c)u^{'}(c)}{\int\limits_{c_0}^{c} \rho(t)[\lambda + V(t)]\, dt} \int\limits_{x}^{c} \frac{1}{\alpha(r)}\, dr \int\limits_{r}^{c} \rho(t)[\lambda + V(t)]\, dt +$$

$$+ \int\limits_{x}^{c} \frac{1}{\alpha(r)}\, dr \int\limits_{r}^{c} \rho(t)[\lambda + V(t)]u(t)\, dt.$$

If we denote

$$M = u(c) - \alpha(c)u^{'}(c) \int\limits_{c_0}^{c} \frac{1}{\alpha(r)}\, dr,$$

$$N = - \frac{\alpha(c)u^{'}(c)}{\int\limits_{c_0}^{c} \rho(t)[\lambda + V(t)]\, dt}$$

and

$$\psi_n(x) = \int\limits_{x}^{c} \frac{1}{\alpha(r_n)}\, dr_n \int\limits_{r_n}^{c} \rho(t_n)[\lambda + V(t_n)]\psi_{n-1}(t_n)\, dt_n, \quad n \geq 1, \quad \psi_0(x) = 1$$

then

$$u(x) \leq M + N\psi_1(x) + \int\limits_{x}^{c} \frac{1}{\alpha(r_1)}\, dr_1 \int\limits_{r_1}^{c} \rho(t_1)[\lambda + V(t_1)]u(t_1)\, dt_1.$$

But

$$u(t_1) \leq M + N\psi_1(t_1) + \int\limits_{t_1}^{c} \frac{1}{\alpha(r_2)}\, dr_2 \int\limits_{r_2}^{c} \rho(t_2)[\lambda + V(t_2)]u(t_2)\, dt_2.$$

By iteration we obtain:

$$u(x) \leq M + N\psi_1(x) + M \int\limits_{x}^{c} \frac{1}{\alpha(r_1)}\, dr_1 \int\limits_{r_1}^{c} \rho(t_1)[\lambda + V(t_1)]\, dt_1 +$$

$$+N\int_x^c \frac{1}{\alpha(r_1)}\,dr_1\int_{r_1}^c \rho(t_1)[\lambda+V(t_1)]\psi_1(t_1)\,dt_1+$$

$$+\int_x^c \frac{1}{\alpha(r_1)}\,dr_1\int_{r_1}^c \rho(t_1)[\lambda+V(t_1)]\,dt_1\int_{t_1}^c \frac{1}{\alpha(r_2)}\,dr_2\int_{r_2}^c \rho(t_2)[\lambda+V(t_2)]u(t_2)\,dt_2\leq$$

$$\leq (M+N)\psi_0(x)+(M+N)\psi_1(x)+N\psi_2(x)+$$

$$+\int_x^c \frac{1}{\alpha(r_1)}\,dr_1\int_{r_1}^c \rho(t_1)[\lambda+V(t_1)]\,dt_1\int_{t_1}^c \frac{1}{\alpha(r_2)}\,dr_2\int_{r_2}^c \rho(t_2)[\lambda+V(t_2)]u(t_2)\,dt_2\leq\cdots$$

$$\cdots\leq (M+N)\sum_{n=0}^{\infty}\psi_n(x).$$

Hence

$$\int_{x_0}^c u(x)\rho(x)\,dx\leq (M+N)\int_{x_0}^c \rho(x)\sum_{n=0}^{\infty}\psi_n(x)\,dx<+\infty.$$

This show the ρ-integrability of u near x_0.

$\Leftarrow$ Assume that (*) and (**) hold. Suppose in contrary that $(\mathcal{A}_1^V, C_0^\infty(x_0,y_0))$ is not $L^\infty(x_0,y_0;\rho dx)$-unique. Then there exists $h\in L^1(x_0,y_0;\rho dx)$, $h\neq 0$ which satisfies

$$\left(\lambda I-(\mathcal{A}_1^V)^*\right)h=0$$

for some $\lambda>0$. We can assume that $h\in C^1(x_0,y_0)$ and $h>0$ on some interval $[x_1,y_1]\subset(x_0,y_0)$, where $x_1<y_1$. Notice that $h^{'}\neq 0$ on (x_1,y_1).

Let $c_1\in(x_1,y_1)$.

(**I**) *case* $h^{'}(c_1)>0$.

By Lemma 3.4, it follows

$$h^{'}(y)>0\quad,\quad\forall y\in(c_1,y_1).$$

Hence

$$h(y)\geq h(c_1)>0\quad,\quad\forall y\in[c_1,y_1].$$

Then we have:

$$h(y) = h(c_1) + \int\limits_{c_1}^{y} h^{'}(r)\, dr =$$

$$= h(c_1) + \int\limits_{c_1}^{y} \left\{ \frac{\alpha(c_1)h^{'}(c_1)}{\alpha(r)} + \frac{1}{\alpha(r)} \int\limits_{c_1}^{r} \rho(t)[\lambda + V(t)]h(t)\, dt \right\} dr >$$

$$> h(c_1) + \int\limits_{c_1}^{y} \frac{1}{\alpha(r)}\, dr \int\limits_{c_1}^{r} \rho(t)[\lambda + V(t)]h(t)\, dt.$$

Using inductively this inequality we get

$$h(y) > h(c_1) + \int\limits_{c_1}^{y} \frac{1}{\alpha(r_1)}\, dr_1 \int\limits_{c_1}^{r_1} \rho(t_1)[\lambda + V(t_1)]h(t_1)\, dt_1 >$$

$$> h(c_1) + h(c_1) \int\limits_{c_1}^{y} \frac{1}{\alpha(r_1)}\, dr_1 \int\limits_{c_1}^{r_1} \rho(t_1)[\lambda + V(t_1)]\, dt_1 +$$

$$+ \int\limits_{c_1}^{y} \frac{1}{\alpha(r_1)}\, dr_1 \int\limits_{c_1}^{r_1} \rho(t_1)[\lambda + V(t_1)]\, dt_1 \int\limits_{c_1}^{t_1} \frac{1}{\alpha(r_2)}\, dr_2 \int\limits_{c_1}^{r_2} \rho(t_2)[\lambda + V(t_2)]h(t_2)\, dt_2 > \cdots$$

$$\cdots > h(c_1) \sum_{n=0}^{\infty} \phi_n(y).$$

Consequently

$$\int\limits_{x_0}^{y_0} h(y)\rho(y)\, dy \geq \int\limits_{c_1}^{y_0} h(y)\rho(y)\, dy > h(c_1) \int\limits_{c_1}^{y_0} \rho(y) \sum_{n=0}^{\infty} \phi_n(y)\, dy = +\infty$$

which is a contradiction with the assumption $h \in L^1(x_0, y_0; \rho dx)$.

(**II**) *case* $h^{'}(c_1) < 0$.

We prove in a similar way that

$$\int\limits_{x_0}^{y_0} h(x)\rho(x)\, dx > +\infty. \quad \square$$

In particular, for $V = 0$, the one-dimensional operator

$$\mathcal{A}_1 f = a(x) f^{''} + b(x) f^{'}$$

is $L^\infty(x_0, y_0; \rho dx)$-unique with respect to the topology $\mathcal{C}(L^\infty(x_0, y_0; \rho dx), L^1(x_0, y_0; \rho dx))$ if an only if both

$$(\circ) \qquad \int_c^{y_0} \rho(y)\, dy \int_c^{y} \frac{1}{\alpha(r)}\, dr \int_c^{r} \rho(t)\, dt = +\infty$$

and

$$(\circ\circ) \qquad \int_{x_0}^{c} \rho(x)\, dx \int_x^{c} \frac{1}{\alpha(r)}\, dr \int_r^{c} \rho(t)\, dt = +\infty$$

hold. In the terminology of Feller this means that y_0 and, respectively x_0 are *no entrance boundaries* (see [30, Theorem 4.1,p.590]).

3.2 The multidimensional case

In this subsection we consider the multidimensional generalized Schrödinger operator

$$\mathcal{A}^V f := \frac{1}{2}\Delta f + b \cdot \nabla f - V f \quad , \quad \forall f \in C_0^\infty(\mathbb{R}^d)$$

where $d \geq 2$ and V is non-negative. Denote the euclidian norm in $\mathbb{R}^d$ by $|x| = \sqrt{x \cdot x}$. If there is some mesurable locally bounded function

$$\beta : \mathbb{R}^+ \to \mathbb{R}$$

such that

$$b(x) \cdot \frac{x}{|x|} \geq \beta(|x|) \quad , \quad \forall x \in \mathbb{R}^d,\ x \neq 0,$$

then for any initial point $x \neq 0$ we have

$$|X_t| - |x| \geq \int_0^t \left[\beta(|X_t|) + \frac{d-1}{2|X_t|}\right] dt + \textit{a real Brownian motion}, \quad \forall t < \tau_e.$$

In other words, $|X_t|$ go to infinity more rapidly than the one-dimensional diffusion generated by

$$\mathcal{A}_1 = \frac{1}{2}\frac{d^2}{dr^2} + \left[\beta(r) + \frac{d-1}{2r}\right]\frac{d}{dr}.$$

This is standard in probability (see IKEDA, WATANABE [13]). Remark that for the one-dimensional operator

$$\mathcal{A}_1^V = \frac{1}{2}\frac{d^2}{dr^2} + \left[\beta(r) + \frac{d-1}{2r}\right]\frac{d}{dr} - V(r)$$

the speed measure of Feller is given by

$$\rho(r) = 2e^{\int_1^r 2\left[\beta(t)+\frac{d-1}{2t}\right]\,dt} = 2e^{\int_1^r 2\beta(t)\,dt}e^{\int_1^r \frac{d-1}{t}\,dt} = 2r^{d-1}e^{\int_1^r 2\beta(t)\,dt}$$

and the scale function of Feller is

$$\alpha(r) = r^{d-1}e^{\int_1^r 2\beta(t)\,dt}.$$

Now we can formulate the main result of this subsection:

THEOREM 3.7. *Suppose that there is some mesurable locally bounded function*

$$\beta : \mathbb{R}^+ \to \mathbb{R}$$

such that

$$b(x)\cdot\frac{x}{|x|} \geq \beta(|x|) \quad , \quad \forall x \in \mathbb{R}^d,\ x \neq 0.$$

If the one-dimensional diffusion operator

$$\mathcal{A}_1^V = \frac{1}{2}\frac{d^2}{dr^2} + \left[\beta(r) + \frac{d-1}{2r}\right]\frac{d}{dr} - V(r)$$

is $L^\infty(0,\infty;\rho dx)$-unique with respect to the topology $\mathcal{C}(L^\infty(0,\infty;\rho dx), L^1(0,\infty;\rho dx))$, then the generalized Schrödinger operator $\left(\mathcal{A}^V, C_0^\infty(\mathbb{R}^d)\right)$ is $L^\infty(\mathbb{R}^d, dx)$-unique with respect to the topology $\mathcal{C}(L^\infty, L^1)$.

Proof. By Theorem 3.2, for the $L^\infty(\mathbb{R}^d, dx)$-uniqueness of $\big(\mathcal{A}^V, C_0^\infty(\mathbb{R}^d)\big)$ it is enough to show that if for some $\lambda > 0$, $u \in L^1(\mathbb{R}^d, dx)$ satisfies

$$\big((\mathcal{A}^V)^* - \lambda I\big)\, u = 0 \quad \textit{in the sense of distributions}$$

then $u = 0$.

Let $\lambda > 0$ and $u \in L^1(\mathbb{R}^d, dx)$ such that

$$\big\langle u, \big(\mathcal{A}^V - I\big)\, f \big\rangle = 0 \quad , \quad \forall f \in C_0^\infty(\mathbb{R}^d)$$

where

$$\langle f, g \rangle := \int_{\mathbb{R}^d} fg\, dx.$$

The above equality becomes

$$\frac{1}{2}\int_{\mathbb{R}^d} u(x)\Delta f(x)\, dx + \int_{\mathbb{R}^d} u(x) b \cdot \nabla f(x)\, dx = \int_{\mathbb{R}^d} u(x)(\lambda + V) f(x)\, dx = 0 \quad , \quad \forall f \in C_0^\infty(\mathbb{R}^d).$$

By the ellipticity regularity result in [9, Lemma 2, p.341], $u \in L^\infty_{loc}(\mathbb{R}^d)$ and $\nabla u \in L^d_{loc}(\mathbb{R}^d) \subset L^2_{loc}(\mathbb{R}^d)$. By the fact that $C_0^\infty(\mathbb{R}^d)$ is dense in

$$\big\{ f \in L^2 \;\big|\; \nabla f \in L^2 \text{ and the support of } f \text{ is compact} \big\}$$

an integration by parts yields

$$-\frac{1}{2}\int_{\mathbb{R}^d} \nabla u(x) \cdot \nabla f(x)\, dx + \int_{\mathbb{R}^d} u(x) b \cdot \nabla f(x)\, dx = \int_{\mathbb{R}^d} u(x)(\lambda + V) f(x)\, dx$$

for all $f \in H^{1,2}(\mathbb{R}^d)$ with compact support. Now on can follow Eberle [9, proof of Theorem 1, 335] to show the next inequality of Kato's type

$$-\frac{1}{2}\int_{\mathbb{R}^d} \nabla |u(x)| \cdot \nabla f(x)\, dx + \int_{\mathbb{R}^d} |u(x)| b \cdot \nabla f(x)\, dx \geq \int_{\mathbb{R}^d} |u(x)|(\lambda + V) f(x)\, dx$$

for all $f \in H^{1,2}(\mathbb{R}^d)$ with compact support.

Let

$$G(r) = \int\limits_{B(r)} |u(x)|\, dx$$

where $B(r) = \left\{ x \in \mathbb{R}^d \,\middle|\, |x| \leq r \right\}$. G is absolutely continuous and

$$G^{'}(r) = \int\limits_{\partial B(r)} |u(x)|\, d_\sigma x \quad , \quad \text{dr-a.e.}$$

where $d_\sigma r$ is the surface measure on the sphere $\partial B(r)$ (the boundary of $B(r)$). Now for every $0 < r_1 < r_2$ we consider

$$f = \min \left\{ r_2 - r_1, (r_2 - |x|)^+ \right\}$$

and

$$\gamma(x) = \frac{x}{|x|} = \nabla|x| \quad .$$

Then we have

$$-\frac{1}{2} \int\limits_{B(r_2)-B(r_1)} \nabla|u(x)| \cdot \nabla(r_2 - |x|)\, dx + \int\limits_{B(r_2)-B(r_1)} |u(x)| b(x) \cdot \nabla(r_2 - |x|)\, dx \geq$$

$$\geq \int\limits_{B(r_2)-B(r_1)} |u(x)|(\lambda + V)(r_2 - |x|)\, dx$$

from where it follows that

$$\frac{1}{2} \int\limits_{B(r_2)-B(r_1)} \nabla|u(x)| \cdot \gamma(x)\, dx - \int\limits_{B(r_2)-B(r_1)} |u(x)| b(x) \cdot \gamma(x)\, dx \geq$$

$$\geq \int\limits_{B(r_2)-B(r_1)} |u(x)|(\lambda + V)(r_2 - |x|)\, dx \quad .$$

Since

$$\nabla|u|\gamma = div(|u|\gamma) - |u| div(\gamma) = div(|u|\gamma) - |u|\frac{d-1}{|x|},$$

by the Gauss-Green formula we have

$$\int\limits_{B(r_2)-B(r_1)} \nabla|u(x)| \cdot \gamma(x)\, dx = G^{'}(r_2) - G^{'}(r_1) - (d-1)\int\limits_{r_1}^{r_2} \frac{1}{r} G^{'}(r)\, dr$$

for $dr_1 \otimes dr_2$-a.e. $0 < r_1 < r_2$.

By another hand, using the hypothese

$$b(x) \cdot \gamma(x) = b(x) \cdot \frac{x}{|x|} \geq \beta(|x|)$$

and Fubini's theorem, we get

$$-\int\limits_{B(r_2)-B(r_1)} |u(x)| b(x) \cdot \gamma(x)\, dx \leq -\int\limits_{r_1}^{r_2} G^{'}(r)\beta(r)\, dr$$

and

$$\int\limits_{B(r_2)-B(r_1)} |u(x)|(\lambda + V)(r_2 - |x|)\, dx = \int\limits_{r_1}^{r_2} [\lambda + V(r)](r_2 - r) G^{'}(r)\, dr =$$

$$= \int\limits_{r_1}^{r_2} [\lambda + V(r)] G^{'}(r) \int\limits_{r}^{r_2} dt\, dr = \int\limits_{r_1}^{r_2} dr \int\limits_{r_1}^{r} [\lambda + V(t)] G^{'}(t)\, dt.$$

Consequently

$$\frac{1}{2}\left[G^{'}(r_2) - G^{'}(r_1)\right] - \int\limits_{r_1}^{r_2} \left[\beta(r) + \frac{d-1}{2r}\right] G^{'}(r)\, dr \geq$$

$$\geq \int\limits_{r_1}^{r_2} dr \int\limits_{r_1}^{r} [\lambda + V(t)] G^{'}(t)\, dt$$

for $dr_1 \otimes dr_2$-a.e. $0 < r_1 < r_2$.

Consider the differential form

$$\mathcal{A}_1^{-} := \frac{1}{2} G^{''}(r) - \left[\beta(r) + \frac{d-1}{2r}\right] G^{'}(r)$$

in the sense of distribution on $(0,\infty)$. Notice that the sign of $\beta(r)+\frac{d-1}{2r}$ in $\mathcal{A}_1^-$ is negative, opposite to the sign in the operator $\mathcal{A}_1^V$ and the speed measure of Feller for $\mathcal{A}_1^-$ is exactely $\rho(r)$ and the scale function of Feller for $\mathcal{A}_1^-$ is $\alpha(r)$. Hence we can write $\mathcal{A}_1^-$ in the Feller form

$$\mathcal{A}_1^- = \frac{1}{2}G'' - \left[\beta(r)+\frac{d-1}{2r}\right]G' = \frac{1}{2}G'' - \frac{\alpha'}{\rho}G' =$$

$$= \frac{1}{2}G'' - \frac{\rho'}{2\rho}G' = \frac{\rho}{2}\frac{\rho G'' - \rho' G'}{\rho^2} = \alpha\left(\frac{G'}{\rho}\right)'.$$

Then we have

$$\left(\frac{G'}{\rho}\right)' \geq \frac{1}{\alpha}\int\limits_{r_1}^{r_2}[\lambda+V(t)]G'(t)\,dt$$

in the sense of distribution on $(0,\infty)$.

Assume now in contrary that $u\neq 0$. Then there exists $c\in(r_1,r_2)$ such that $G'(c)>0$. Then for dy-a.e. $y>c$ we have

$$\frac{G'}{\rho}(y) \geq \frac{G'}{\rho}(c) + \int\limits_c^y \frac{1}{\alpha(r)}\,dr\int\limits_c^r[\lambda+V(t)]G'(t)\,dt =$$

$$= \frac{G'}{\rho}(c) + \int\limits_c^y \frac{1}{\alpha(r)}\,dr\int\limits_c^r \rho(t)[\lambda+V(t)]\frac{G'}{\rho}(t)\,dt.$$

Using the above inequality inductively we get

$$\frac{C'}{\rho}(y) \geq \frac{G'}{\rho}(c)\sum_{n=0}^{\infty}\phi_n(y)$$

where $\phi_0(y)=1$ and for any $n\in\mathbb{N}^*$,

$$\phi_n(y) = \int\limits_c^y \frac{1}{\alpha(r_n)}\,dr_n\int\limits_c^{r_n}\rho(t_n)[\lambda+V(t_n)]\phi_{n-1}(t_n)\,dt_n.$$

By Theorem 3.7 it follows that

$$\int\limits_{\mathbb{R}^d} |u(x)|\, dx = G(\infty) \geq \frac{G'}{\rho}(c) \int\limits_{c}^{\infty} \rho(y) \sum_{n=0}^{\infty} \phi_n(y)\, dy = +\infty$$

because $\mathcal{A}_1^V$ is suppose to be $L^\infty(0,\infty;\rho dx)$-unique. This in contradiction with the assumption that $u \in L^1(\mathbb{R}^d, dx)$. $\square$

Remark that if $\mathcal{A}$ is a second order elliptic differential operator with $\mathcal{D} = C_0^\infty(\mathbb{R}^d)$, then the weak solutions for the dual Cauchy problem in the Theorem 3.2 (v) correspond exactly to those in the distribution sense in the theory of partial differential equations and the dual Cauchy problem becomes the Fokker-Planck equation for heat diffusion. Then we can formulate

COROLLARY 3.8. *In the hypothesis of Theorem 3.7, for any $f \in L^1(\mathbb{R}^d, dx)$ the Fokker-Planck equation*

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) - div\,(bu(t,x)) - Vu(t,x) \\ u(0,x) = f(x) \end{cases}$$

has one $L^1(\mathbb{R}^d, dx)$-unique weak solution.

Proof. The assertion follows by the Theorem 3.2 and the Theorem 3.7. $\square$

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